Geometric And Variational Analysis.
In memory of Jan Malý.
9th June-15th June 2024, Będlewo, Poland

Plenary talks

# Monday

**Stanislav Hencl (Charles University)**
*Weak limit of homeomorphisms in $W^{1,n-1}$ and (INV) condition*

Let $\Omega, \Omega' \subset \mathbb{R}^3$ be Lipschitz domains, let $f_m : \Omega \to \Omega'$ be a sequence of homeomorphisms with prescribed Dirichlet boundary condition and $\sup_m \int_\Omega (|Df_m|^2 + 1/J_{f_m}^2) < \infty$. Let $f$ be a weak limit of $f_m$ in $W^{1,2}$. We show that $f$ is invertible a.e., more precisely it satisfies the (INV) condition of Conti and De Lellis and thus it has all the nice properties of mappings in this class.

Generalization to higher dimensions, an example showing sharpness of the condition $1/J_f^2 \in L^1$ and application to existence of minimizers in the class of weak limits of homeomorphisms are also given. Using this example we also show that unlike the planar case the class of weak limits and the class of strong limits of $W^{1,2}$ Sobolev homeomorphisms in $\mathbb{R}^3$ are not the same.

This is a joint result with A. Doležalová, J. Malý and A. Molchanova.

**Irene Fonseca (Carnegie Mellon University)**
*From Phase Separation in Heterogeneous Media to Learning Training Schemes for Image Denoising*

What do these two themes have in common? Both are treated variationally, both deal with energies of different dimensionalities, concepts of geometric measure theory prevail in both, and higher order penalizations are considered. Will learning training schemes for choosing these penalizations in imaging may be of use in phase transitions?

**Phase Separation in Heterogeneous Media**: Modern technologies and biological systems, such as temperature-responsive polymers and lipid rafts, take advantage of engineered inclusions, or natural heterogeneities of the medium, to obtain novel composite materials with specific physical properties. To model such situations using a variational approach based on the gradient theory of phase transitions, the potential and the wells may have to depend on the spatial position, even in a discontinuous way, and different regimes should be considered. In the critical case case where the scale of the small heterogeneities is of the same order of the scale governing the phase transition and the wells are fixed, the interaction between homogenization and the phase transitions process leads to an anisotropic interfacial energy. The supercritical case for fixed wells is also addressed, now leading to an isotropic interfacial energy. In the subcritical case with moving wells, where the heterogeneities of the material are of a larger scale than that of the diffuse interface between different phases, it is observed that there is no macroscopic phase separation and that thermal fluctuations play a role in the formation of nanodomains. This is joint work with Riccardo Cristoferi (Radboud University, The Netherlands) and Likhit Ganedi (Aachen University, Germany), USA), based on previous results also obtained with Adrian Hagerty (USA) and Cristina Popovici (USA).

**Learning Training Schemes for Image Denoising**: Due to their ability to handle discontinuous images while having a well-understood behavior, regularizations with total variation (TV) and total generalized variation (TGV) are some of the best known methods in image denoising. However, like other variational models including a fidelity term, they crucially depend on the choice of their tuning parameters. A remedy is to choose these automatically through multilevel approaches, for example by optimizing performance on noisy/clean image training pairs. Such methods with space-dependent parameters which are piecewise constant on dyadic grids are considered, with the grid itself being part of the minimization.

Existence of minimizers for discontinuous parameters is established, and it is shown that box constraints for the values of the parameters lead to existence of finite optimal partitions. Improved performance on some representative test images when compared with constant optimized parameters is demonstrated. This is joint work with Elisa Davoli (TU Wien, Austria), Jose Iglesias (U. Twente, The Netherlands) and Rita Ferreira (KAUST, Saudi Arabia)

**Piotr Hajłasz (University of Pittsburgh)**
*Lusin approximation of convex functions*

My talk is based on three recent papers listed below.

It has been known for at least thirty years, that convex functions have the $C^2$-*Lusin property*, meaning that if $u:\mathbb{R}^n\to\mathbb{R}$ is convex, then for every $\varepsilon>0$, there is a function $v\in C^2(\mathbb{R}^n)$ such that $|\{u\neq v\}|<\varepsilon$. However, in general, $v$ cannot be convex: *If* $u:\mathbb{R}\to\mathbb{R}$, $u(x,y)=|x|$, *then the only convex function* $v:\mathbb{R}^2\to\mathbb{R}$ *such that* $|\{u\neq v\}|<\infty$, *is* $v=u$. Thus one may ask: *When can we find such convex* $v$*?*

The problem of approximating a convex function by $C^2$-convex functions in the Lusin sense has remained unresolved since the nineties. It was natural to expect a positive answer in the case of *strongly convex functions* or, more generally, *locally strongly convex functions.*

A function $u$ is *strongly convex*, if there exists $\eta>0$ such that $u(x)-\frac{\eta}{2}|x|^2$ is convex. In this case, we say that $u$ is $\eta$*-strongly convex.* Moreover, $u$ is *locally strongly convex* whenever, for every $x$, there exists $r_x>0$ such that the restriction of $u$ to $B(x,r_x)$ is strongly convex. In the recent paper [1] we answered this question in the positive:

**Theorem** ([1]). *Let* $U\subseteq\mathbb{R}^n$ *be open and convex, and* $u:U\to\mathbb{R}$ *be locally strongly convex. Then for every* $\varepsilon_o>0$ *and for every continuous function* $\varepsilon:U\to(0,1]$ *there is a locally strongly convex function* $v\in C^2(U)$, *such that*

*(a)* $|\{x\in U:\, u(x)\neq v(x)\}|<\varepsilon_o$;

*(b)* $|u(x)-v(x)|<\varepsilon(x)$ *for all* $x\in U$.

*Also, if* $u$ *is* $\eta$*-strongly convex on* $U$, *then for every* $\widetilde{\eta}\in(0,\eta)$ *there exists such a function* $v$ *which is* $\widetilde{\eta}$*-strongly convex on* $U$.

An earlier result published in [3] answers the problem of Lusin approximation by $C^{1,1}$ convex functions (functions with the Lipschitz continuous gradient).

**Theorem** ([3]). *If* $u:\mathbb{R}^n\to\mathbb{R}$ *is convex and coercive* ($f(x)\to\infty$ *as* $|x|\to\infty$), *then for every* $\varepsilon>0$, *there is a convex function* $v\in C^{1,1}_{\rm loc}(\mathbb{R}^n)$ *such that* $|\{x\in\mathbb{R}^n:u(x)\neq v(x)\}|<\varepsilon$.

The paper [2] contains a much simpler proof (and some extensions) of this result, and it shows how to use it to provide a simple proof of the Alexandrov theorem about the second order differentiability of convex functions.


[1] D. Azagra, M. Drake, P. Hajłasz, $C^2$-Lusin approximation of strongly convex functions. *Inventiones Math.* (Published online)

[2] D. Azagra, A. Cappello, P. Hajłasz, A geometric approach to second-order differentiability of convex functions. *Proc. Amer. Math. Soc. Ser. B* 10 (2023), 382–397.

[3] D. Azagra, P. Hajłasz, Lusin-type properties of convex functions and convex sets. *J. Geom. Anal.* 31 (2021), 11685–11701.


**Pekka Koskela (University of Jyväskylä)**
*Quasiconformal mappings and borderline Sobolev spaces*

The borderline first order homogeneous Sobolev space consisting of those functions whose first order distributional partial derivatives are integrable at the level of the dimension of the underlying space is preserved under global quasiconformal changes of variable. The same conclusion holds for the homogeneous fractional Sobolev spaces when the product of the order of derivatives with the integrability degree coincides with the underlying dimension. In the case of the usual Sobolev setting, the invariance also holds in the case of domains and quasiconformal mappings defined on them. This does not extend as such to the fractional setting. I will describe recent attempts to understand the fractional case.

The results are from joint works with Nijjwal Karak, Kaushik Mohanta, Debanjan Nandi and Swadesh Sahoo.

**Jani Onninen (Syracuse University)**
*Quasiregular values*

Quasiregular maps form a higher-dimensional class of maps with many similar properties to holomorphic maps, such as continuity, openness, discreteness, and versions of the Liouville and Picard theorems. In this talk, we give a pointwise definition of quasiregularity. We show that this condition yields counterparts to many fundamental properties of quasiregular maps at a single point. The studied maps have already shown to play a key part in various important 2D results. Joint work with Ilmari Kangasniemi.

**Tadeusz Iwaniec (Syracuse University)**
*Sobolev Mappings and Energy Integrals in Geometric Function Theory and Nonlinear Hyper-Elasticity*

As we seek greater knowledge about the energy-minimal deformations the questions about Sobolev homeomorphisms and their limits become ever more quintessential. In his models of Nonlinear Elasticity, Sir John Ball formulated the Principle of Non-Interpenetration of Matter, which requires that hyper-elastic deformations be energetically impossible to compress part of the body of positive volume into zero volume. Essentially, he required that the Jacobian determinants be positive. However, from a mathematical point of view, one quickly runs into a serious difficulty when passing to a weak limit of an energy-minimizing sequence of Sobolev homeomorphisms (Direct Method in the Calculus of Variations does not guarantee such a principle). Injectivity can be lost due to squeezing of the material. We say that "squeezing interpenetration of matter" may occur (though folding turns out to be impossible). In planar domains, such limits are topologically identified as Monotone Sobolev Mappings. It is at the heart of the present lecture to convince the listeners that the limits of Sobolev Mappings in every dimension, having particular topological characteristics, should be legitimate deformations in mathematical models of Hyper-Elasticity.

**Olli Martio (University of Helsinki)**
*Capacities from moduli in metric measure spaces*

It is shown that in metric measure spaces the $AM_p(\Gamma)$– and $M_p(\Gamma)$–modulus create the capacities $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{AM}}(E,G)$ and $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G)$, respectively where $\Gamma$ is the path family connecting an arbitrary set $E \subset G$ to the complement of a bounded open set $G$. The construction uses Lipschitz functions and their upper gradients. For $p > 1$ the capacities coincide but differ for $p = 1$. For $p \geq 1$ it is shown that the $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{AM}}(E,G)$–capacity equals to the classical variational Dirichlet capacity of the condenser $(E,G)$, the $\mathrm{Cap}_{\mathrm{p}}^{\mathrm{M}}(E,G)$–capacity to the $M_p(\Gamma)$–modulus and $AM_1(\Gamma(E)) \leq Cap_1^{AM}(E,G) \leq M_1(\Gamma(E))$.

# Tuesday

**John Ball (Heriot-Watt University and Maxwell Institute for Mathematical Sciences)**
*Image comparison and scaling via nonlinear elasticity*

A nonlinear elasticity model for comparing images is formulated and analyzed, in which optimal transformations between images are sought as minimizers of an integral functional. The existence of minimizers in a suitable class of homeomorphisms between image domains is established under natural hypotheses, and the question of whether for linearly related images the minimization algorithm delivers the linear transformation as the unique minimizer is discussed. This is joint work with Chris Horner.

**Duvan Henao (Universidad de O'Higgins)**
*Harmonic dipoles in nonlinear elasticity*

Malý (1993) proved that the relaxation of the neoHookean energy coincides with the neo-Hookean energy at diffeomorphisms, thus establishing the first existence result for neoHookean materials in 3D. In joint work with Barchiesi, Mora-Corral & Rodiac [ARMA 247 (2023); FoM:Sigma 12 (2024)], we present some progress on the more explicit understanding of which deformations can fall into the weak closure of regular (injective, orientation-preserving, controlled Jacobian) maps and of the relaxed energy evaluated at deformations with singularities. For the pathological example of Conti & De Lellis (2003) we show that the singular energy is precisely twice the length of the dipole times the area of the bubble across which two portions of the elastic body which were separated in the reference configuration are now in contact in the deformed configuration. This, in turn, coincides with twice the total variation of the singular part of the derivative of the inverse map. We show that in the weak closure all maps have inverses with BV regularity, and in the axisymmetric case establish the Sobolev regularity for the first two components. In this axisymmetric case we obtain, as a lower bound for the singular energy, precisely twice the variation of the singular part of the inverse. In the case of maps which furthermore possess SBV regularity for the inverse, we show that the singularities are dipoles, showing that the example of Conti & De Lellis is very generic.

**Carlos Mora-Corral (Universidad Autónoma de Madrid)**
*Variational models in materials with Eulerian and Lagrangian energies*

We show variational models in material science consisting in the minimization of an energy with two parts: one defined in the reference configuration (Lagrangian formulation) and the other in the deformed configuration (Eulerian formulation). These models describe the coupling of elastic deformations with other effects. Typical examples arise in the theory of nematic elastomers or magnetoelasticity, and the energies take the form

$$\int_\Omega W(Du(x), w(u(x)))dx + \int_{u(\Omega)} |Dw(y)|^2 dy + \text{other terms.}$$

Here $\Omega \subset \mathbb{R}^n$ is the reference configuration, $u : \Omega \to \mathbb{R}^n$ is the deformation, and $w : u(\Omega) \to \mathbb{R}^m$ is an Eulerian field describing, for example, the direction of the nematic elastomer or the magnetization, depending on the model. Two key difficulties in the analysis are the convergence of the images $u_j(\Omega)$ and of compositions $w_j \circ u_j$ under weak convergence. We present existence results for an ample family of such energies. We divide those results according to the regularity assumptions on $u$: when $u$ is purely elastic, when $u$ presents cavitation, and when $u$ can have cracks. This is a joint work with M. Barchiesi, M. Bresciani, D. Henao and M. Friedrich.

# Wednesday

**Luigi Ambrosio (SNS Pisa)**

*Sharp PDE estimates for random two-dimensional bipartite matching with power cost function*

We investigate the random bipartite optimal matching problem on a flat torus in two-dimensions, considering general strictly convex power costs of the distance. We extend the successful ansatz first introduced by Caracciolo, Parisi et al. for the quadratic case, involving a linear Poisson equation, to a non-linear equation of $q$-Poisson type, allowing for a more comprehensive analysis of the optimal transport cost. Our results establish new asymptotic connections between the energy of the solution to the PDE and the optimal transport cost, providing insights on their asymptotic behavior.

**Aldo Pratelli (University of Pisa)**

*On the connectedness properties of minimal clusters of small volume in Riemannian of Finsler manifolds*

It is known that isoperimetric sets of small volume in a Riemannian manifold are close to balls, since the manifold locally looks like $\mathbb{R}^N$. The situation for minimal clusters is less known. We will prove that, in a compact Riemannian manifold small minimal clusters must be also connected. More in general, in a compact Finsler manifold we can prove that small minimal $m$-clusters (that is, clusters with $m$ chambers) contain at most m connected components. Even though it seems reasonable that minimal clusters are connected also in the Finsler case, we will prove through an explicit example that it is not so (the results are based on joint works with D. Carazzato and S. Nardulli).

**Agnieszka Kałamajska (University of Warsaw)**

*Poincaré inequalities and compact embeddings from Sobolev type spaces into weighted $L^q$ spaces on metric spaces*

We study compactness and boundedness of embeddings from Sobolev type spaces on metric spaces into $L^q$ spaces with respect to another measure. The considered Sobolev spaces can be of fractional order and the involved measures are not assumed to be doubling. Our results are using sequences of covering families and exploit local Poincaré type inequalities. We show how to construct such suitable coverings and Poincaré inequalities and prove a self-improvement property for two-weighted Poincaré inequalities. We simultaneously treat various Sobolev and Slobodecki type spaces, including Newtonian, fractional Hajłasz and Sobolev–Slobodetskiĭ spaces in the metric setting. As the special case we also obtain trace embeddings for the above spaces. In the case of Newtonian spaces we characterize when embeddings into $L^q$ spaces with respect to another measure are compact.

The talk is based on joint work with Jana Björn [Journal of Functional Analysis 282(11), 2022].

# Thursday

**Giuseppe Mingione (University of Parma)**
*Nonuniformly elliptic Schauder theory*

Schauder estimates are a basic tool in elliptic and parabolic PDE. The idea is to show that solutions are as regular as coefficients allow. They serve as a basic tool in a wide variety of situations: higher regularity of solutions to problems showing any kind of ellipticity, including free boundaries, bootstrap processes, existence theorems and so on. Their validity in the setting of linear uniformly elliptic problems is classical. First results were obtained by Hopf, Giraud, Caccioppoli and Schauder in the 20/30s of the past century. Extensions were obtained by Agmon, Douglis and Nirenberg. New proofs were achieved over the years by Campanato, Trudinger, Simon (via suitable function spaces, convolution, blow-up, respectively). Nonlinear versions were achieved by Giaquinta & Giusti, DiBenedetto, Manfredi. More recently, nonlocal versions were obtained as well. As the equations in question are non-differentiable, all these approaches unavoidably rely on perturbation methods, i.e., freezing coefficients and comparing original solutions to solutions with constant coefficients problems. Such approaches, relying on the availability of homogenous estimates for frozen problems, ceases to work in nonuniformly elliptic problems, for which such homogeneity is lost, and for which the validity of Schauder theory has remained an open problem for decades. We shall present a full solution to the problem of Schauder estimates in the nonlinear, nonuniformly elliptic setting. In particular, we shall present the first direct, non-perturbative approach to pointwise gradient estimates for non-differentiable equations ever. From recent, joint work with Cristiana De Filippis (Parma).

**Jan Kristensen (University of Oxford)**
*The Burkholder functional on classes of planar maps*

The area formula of Gronwall and Bieberbach can be viewed as a precise way to express that the Jacobian is a null Lagrangian. In this talk I discuss a quasiconvexity inequality for the Burkholder functional in the context of planar quasiconformal maps. This inequality can be viewed as an extention of the area formula to an $L_p$ context. If combined with Stoilow factorization and blow-up arguments it also allows a proof of semicontinuity, and hence to prove existence of minimizers for the Burkholder energy on classes of planar quasiregular maps. The talk is based on joint work with Kari Astala (Helsinki), Daniel Faraco (Madrid), Andre Guerra (ETH), and Aleksis Koski (Aalto).

**Andre Guerra (ETH Zurich)**
*Unique continuation, Quasiconformal Maps, and the Monge–Ampère Equation*

In the complex plane, there is a surprising correspondence between solutions of the Monge–Ampère equation and solutions of a certain differential inclusion associated to SO(2). Under this correspondence, the $W^{2,p}$ regularity of solutions to Monge–Ampère corresponds to a precise unique continuation principle for solutions of the differential inclusion. We will sketch a proof of the latter, which relies on quasiconformal maps and the rigidity estimate for SO(2). This proof also yields new results for nonlinear Beltrami equations in the plane. The talk is based on joint work with G. De Philippis and R. Tione.

# Friday

**Monica Torres (Purdue University)**
*Cauchy fluxes and extended divergence-measure fields*

We extend the balance law from classical continuum physics to the case where the production on any open set is measured with a Radon measure, and the associated Cauchy flux is bounded by a Radon measure concentrated on the boundary of the set. We prove that there exists an extended divergence-measure field (i.e.; a vector-valued Radon measure whose distributional divergence is also a Radon measure) such that the Cauchy flux can be recovered through the field, locally on almost every open set, and globally for every open set. We establish a one-to-one correspondence between Cauchy fluxes and extended divergence-measure fields. Our results generalize the classical Cauchy's theorem which was only valid for continuous vector fields, and extends previous formulations of the Cauchy flux that generated vector fields in $L^p$. This is a joint work with Gui-Qiang Chen (University of Oxford) and Christopher Irving (Technical University Dortmund).

**Luigi D'Onofrio (University of Napoli "Parthenope")**
*On $BV$ homeomorphisms*

We obtain the rectifiability of the graph of a bounded variation homeomorphism $f$ in the plane and relations between gradients of $f$ and its inverse. Further, we show an example of a bounded variation homeomorphism $f$ in the plane which satisfies the Lusin property and strict positivity of Jacobian of both itself and its inverse is Sobolev.

**Luboš Pick (Charles University)**
*Potential trace inequalities via a Calderón-type theorem*

This contribution is dedicated to the memory of my dear friend Jan Malý. The talk will begin with a short personal recollection of good times spent together.

The main aim of the scientific part of the talk will be to describe a general technique which was recently developed jointly with Zdeněk Mihula and Daniel Spector, and which had been directly inspired by the limiting version of the trace inequality for the Riesz potential obtained earlier by Mikhail Korobkov and Jan Kristensen, and also by some discussions with Jan Malý which were carried out right after the paper of Korobkov and Kristensen had appeared.

We develop a general technique that enables us to establish boundedness results for certain 'bad' operators (modeled upon a pivotal example of the Riesz potential) from interpolation properties of suitable 'good' operators (such as the fractional maximal operator) from a classical Lorentz space with summation property into a specific rearrangement-invariant function space $X^{\langle p \rangle}$, where $X$ is an appropriate rearrangement-invariant space and $p \in [1, \infty)$. These spaces appeared recently in connection with Sobolev embeddings into spaces endowed with Frostman measures in a joint work with Andrea Cianchi and Lenka Slavíková, and were later studied in detail by Hana Turčinová. The approach is based on a combination of Sawyer's duality theorem with a result in spirit of the classical interpolation theorem by Calderón, suitably adapted to the situation at hand. Among possible applications of the general scheme we obtain a new proof of the trace inequality for the Riesz potential by Korobkov and Kristensen.

The talk is based on a manuscript written jointly with Zdeněk Mihula (FEE CTU Prague) and Daniel Spector (NTNU Taipei) which has not been published yet.